Non-analyticity in the distribution of conductances in quasi one dimensional wires

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Abstract. – We show that the distribution $P(g)$ of conductances g of a quasi one dimensional wire has non-analytic behavior in the insulating region, leading to a discontinuous derivative in the distribution near $g = 1$. We give analytic expressions for the full distribution and extract an approximate scaling behavior valid for different strengths of disorder close to $g = 1$.

It is well known that the probability distribution $P(g)$ of the (dimensionless) conductances g of a disordered conductor is Gaussian in the deeply metallic regime and log-normal in the deeply insulating regime [1]. It has been proposed recently [2] that for a quasi one dimensional (1D) wire with mean dimensionless conductance $\bar{g} \ll 1$, $P(g)$ has a sharp cut-off beyond $g = 1$. To be sure, in one dimension, when $g \leq 1$, $P(g)$ drops discontinuously to zero at $g = 1$. In higher dimensions the discontinuity will be smeared in some way, but it is difficult to say how much of it will survive. Using the saddle point method developed in [2], we are able to show here that nonanalytic behavior remains near $g = 1$ at least in quasi-1D systems. One should note that the exact results for the mean and variance of the conductance, obtained within the non-linear sigma model [3], do not give any clue on the abrupt change of $P(g)$ near $g = 1$. The existence of the sharp cutoff for all $\bar{g} \ll 1$ has recently been confirmed numerically, while the exact shape of the distribution very close to $g = 1$ seems to be different from a log-normal distribution [4, 6, 7, 5, 8].

In the present work, we focus on the insulating region in quasi 1D near $|g - 1| \ll 1$, in order to understand better the nature of the unexpected sharp feature in the distribution. For simplicity, we will restrict our discussions to the unitary case of broken time reversal symmetry. We propose from a simple generalization of [2] that for large enough disorder the distribution has a non analyticity near $g = 1$, giving rise to large discontinuities in its derivatives. We obtain e.g. that in the insulating limit, the leading contribution to the distribution $P(g)$ at

$|g-1| \ll 1$, is given by

$$P(g) \propto \begin{cases} \Phi\left(\sqrt{\Gamma}(\nu - \frac{1}{2\Gamma})\right) - \Phi\left(\sqrt{\Gamma}(\nu_1 - \frac{1}{2\Gamma})\right) + C; & g \geq 1+\alpha, \\ \Phi\left(\frac{1}{\sqrt{\Gamma}})\right) - \Phi\left(\sqrt{\Gamma}(\nu_1 - \frac{1}{2\Gamma})\right) + C; & g < 1+\alpha, \end{cases} \tag{1}$$

where Φ is the error function, $\nu = \cosh^{-1}\sqrt{1/(g-1)}$, $\nu_1 = \cosh^{-1}\sqrt{2/g}$, and $\alpha = 1/\cosh^2(3/2\Gamma)$ [9]. The disorder parameter $\Gamma = \xi/L$, where ξ is the localization length and L is the system size ($\Gamma \gg 1$ would correspond to metals and $\Gamma \ll 1$ corresponds to insulators). The term C is independent of g, but depends on Γ. The discontinuity is at $g = 1+\alpha$. For the case considered here, $\Gamma \ll 1$ and $\alpha \sim e^{-3/\Gamma}$. From (1) it follows that $P' = \mathrm{d}P(g)/\mathrm{d}g$ has a discontinuity at $g = 1+\alpha$, with $P' \sim -e^{2/\Gamma}$ very large for $g \gtrsim 1+\alpha$ (growing exponentially with increasing disorder) and $|P'| \ll 1$ for $g < 1+\alpha$. Note that in the limit $\Gamma \to 0$, the distribution would have an essential singularity at $g = 1$.

The above results are obtained assuming $\Gamma \ll 1$, describing the insulating limit. As Γ increases, but still in the insulating regime, the dominant contribution remains the same with renormalized parameters, and our results remain qualitatively valid. In the metallic limit the singularity is absent, giving rise to the possibility that the singularity disappears across the crossover region ($\Gamma \sim 1/2$), where our current approximations are not valid.

In addition to the sharp structure, our results suggest an approximate scaling behavior for different strengths of disorder near $g \sim 1$. As is evident from (1), in the expression for the ratio $P(g)/P(1)$, the dominant dependence on the disorder parameter Γ and on $g-1$ appears only in the combination $\sqrt{\Gamma}(\nu - 1/2\Gamma)$. Our numerical results agree with this approximate scaling. We also find that the distribution $P(g)$ for $g \lesssim 1$ is a slowly varying function of g, which is approximately constant close to $g = 1$. This leads to an exponential distribution for $P(\ln g)$ in this region, rather than the log-normal distribution proposed in [2], and agrees with various existing numerical results [6, 4]. For $\Gamma \ll 1$, the numerical results suggest that the exponential form crosses over to the log-normal form for $g \ll 1$.

We now briefly discuss the details of the method which is based on a simple generalization of [2]. The probability distribution $p(\lambda)$ of the N variables λ_i, where λ_i are related to the transmission eigenvalues T_i of an N-channel quasi 1d wire by $\lambda_i = (1-T_i)/T_i$, satisfy the well known DMPK equation [10], whose solutions in the metallic and insulating regimes can be written in the general form [11]

$$p(\lambda) = \frac{1}{Z}\exp[-\beta H(\lambda)], \tag{2}$$

where $Z = \int \prod_i \mathrm{d}\lambda_i \exp[-\beta H]$ is a normalizing factor independent of λ_i, $H(\lambda)$ may be interpreted as the Hamiltonian function of N classical charges at positions λ_i, and $\beta = 2$ for the unitary case. Since the dimensionless conductance is given by

$$g = \sum_i^N \frac{1}{1+\lambda_i}, \tag{3}$$

the probability distribution $P(g)$ can be written as

$$P(g) = \frac{1}{Z}\int_{-\infty}^{\infty}\frac{\mathrm{d}\tau}{2\pi}\int_0^{\infty}\prod_{i=1}^{N}\mathrm{d}\lambda_i \exp\left[i\tau(g - \sum_i^N \frac{1}{1+\lambda_i})\right] p(\lambda). \tag{4}$$

Following [2], we define a "Free energy"

$$F(\lambda) = \sum_{i}^{N} \frac{i\tau}{1+\lambda_i} + \beta H \tag{5}$$

such that the distribution can be written as

$$P(g) = \frac{1}{Z} \int_{-\infty}^{\infty} \frac{\mathrm{d}\tau}{2\pi} e^{i\tau g} \int_{0}^{\infty} \prod_{i=1}^{N} \mathrm{d}\lambda_i \exp\left[-F(\lambda)\right]. \tag{6}$$

It has been shown in [2] that separating out the lowest eigenvalue λ_1 and treating the rest of the eigenvalues as a continuum beginning at λ_2, the distribution can be obtained within a generalized saddle point approximation, given by

$$P(g) = \int_{0}^{\infty} \mathrm{d}\lambda_1 \int_{\lambda_1}^{\infty} \mathrm{d}\lambda_2 e^{-S}, \tag{7}$$

where

$$S = -\frac{1}{2F''}(g - F')^2 + F^0 \tag{8}$$

is the saddle point action, obtained from a saddle point free energy

$$F_{sp} = F^0 + (i\tau)F' + \frac{(i\tau)^2}{2}F''. \tag{9}$$

The insulating region in this model is given by $x_2 \gg 1$, and also $x_2 \gg x_1$, where $\sinh^2 x_i = \lambda_i$. The saddle point free energy terms in this limit were calculated to be [12]

$$F^0(x_1, x_2) \approx 2\Gamma^2 x_2^2 - 6\Gamma x_2 + \Gamma x_1^2 - \frac{1}{2}\ln(x_1 \sinh(2x_1)); \tag{10}$$

$$F' \approx \frac{1}{\cosh^2 x_1}; \tag{11}$$

$$F'' = -\frac{1}{\sinh^2(2x_2)}\left[\frac{1}{3} - \frac{1}{4x_2^2} + \frac{1}{\sinh^2(2x_2)}\right]. \tag{12}$$

Since $1/2|F''|$ is exponentially large in the insulating regime, the saddle point solution is given by putting the coefficient of the term in the action S equal to zero, namely $g = F'$, which then gives

$$\cosh x_{1sp} = \frac{1}{\sqrt{g}}. \tag{13}$$

As pointed out in [2], the saddlepoint solution is valid only for $g < 1$ since $\cosh x_1 \geq 1$, while the $g > 1$ region is determined by the boundary values of $x_1 = 0$ and $x_2 = 2/\pi\Gamma$. While this is a good approximation on both sides of $g = 1$, the fact that the boundary of the saddle point solution is at $g = 1$ makes it possible that the approximation is not accurate enough very close to $g = 1$. We will show below that at the next level of approximation suggested in [2], the region close to $g = 1$ can be better described by saddle point solutions valid on both sides. This improved approximation immediately leads to the non-analyticity mentioned before.

It was shown in [13] that while separating out the lowest eigenvalue gave qualitatively good results both in the metallic and insulating regimes, it is important to separate out one

additional eigenvalue to obtain good agreement with numerical and available exact results in the insulating and crossover regimes because the separation between the eigenvalues becomes large. The extension is straightforward, and the distribution $P(g)$ now has an additional integral over λ_3, which is the beginning of the continuum that represents the rest of the eigenvalues. It is clear that the insulating limit is characterized by $x_3 \gg x_2 \gg 1$ for typical values of x_i, and that F' in this limit will now be given by [13]

$$F' \approx \frac{1}{\cosh^2 x_1} + \frac{1}{\cosh^2 x_2}. \tag{14}$$

F'' now has the same expression as (12), with x_2 replaced by x_3, while F^0 is given by (10) with x_1, x_2 replaced by x_2, x_3 plus additional terms involving x_1

$$F^0(x_1, x_2, x_3) = F^0(x_2, x_3) + \Gamma x_1^2 - \frac{1}{2}\ln(x_1 \sinh(2x_1)). \tag{15}$$

The saddle point solutions for x_3 and x_2 are now given by $x_{3sp} = 3/2\Gamma$, which is independent of g and

$$\cosh x_{2sp} = \frac{1}{\sqrt{g - 1/\cosh^2 x_1}}, \tag{16}$$

obtained again from $g = F'$. Since $x_2 \gg 1$, the additional term in F' is exponentially small, and usually negligible. However, close to $g = 1$, this is the term that allows the saddle point solution to be valid on both sides of $g = 1$. The fluctuation correction to the x_2 integral contributes an additional factor $e^{2x_{2sp}}$, and we are left with the final integral

$$P(g) \propto \int_{x_{1min}}^{x_{1max}} \mathrm{d}x_1 e^{-\Gamma x_1^2 + \ln x_1 - \Gamma x_{2sp}^2 + 3x_{2sp}}. \tag{17}$$

We have used finite limits in the integral because the saddle point solution of x_2 must be real and also we require that $x_{3sp} > x_{2sp}$. This gives $x_{1min} = Re[\cosh^{-1}\sqrt{1/(g-\alpha)}]$, $\alpha = 1/\cosh^2(3/2\Gamma)$, and $x_{1max} = \cosh^{-1}\sqrt{2/g}$. The crucial point is that the restriction x_{1min} is zero for all $g \geq 1 + \alpha$, but becomes finite for $g < 1 + \alpha$, with a discontinuous slope. *It is the universal nature of this constraint which will eventually lead to the sharp changes in $P(g)$ at $g = 1 + \alpha$.*

In the limit $g \to 1$ and $\Gamma \ll 1$, the upper limit of the integral $x_{1max} \ll 1$. We can therefore neglect the x_1^2 term in (17). Changing variables to x_2 and rewriting $\ln x_1$, we obtain a simple integral

$$P(g) \propto \int_{\nu_1}^{x_{2max}} \mathrm{d}x_2 e^{-\Gamma x_2^2 + x_2} \tag{18}$$

with ν_1 as defined after (1), and $x_{2max} = \nu$ for $g > 1 + \alpha$, but $3/2\Gamma$ for $g < 1 + \alpha$. The integral immediately leads to Eq. (1). It is easy to see from (1) that the first derivative has a discontinuity at $g = 1 + \alpha$ and is large, of order $e^{2/\Gamma}$ for $g > 1 + \alpha$. Figure 1 shows the sharp changes at $g = 1 + \alpha$ according to Eq. (1). Since ν_1 in this region is only weakly dependent on g, this also leads to an approximate scaling behavior of $P(g)$ as a function of $z = \sqrt{\Gamma}(\nu - 1/2\Gamma)$.

While the g-dependent contribution comes from the x_2 integral where $x_{2sp} < x_{3sp}$, there is a contribution to $P(g)$ from integral over $x_2 > x_{3sp}$. The x_3 integral in this case is given by the boundary value at $x_3 = x_2$, and the fluctuation correction to the x_2 integral becomes of order unity. The resulting integral leads to a g independent but Γ dependent term giving $C \approx e^{3/\Gamma}[1 - \Phi(2/\sqrt{\Gamma})]$.

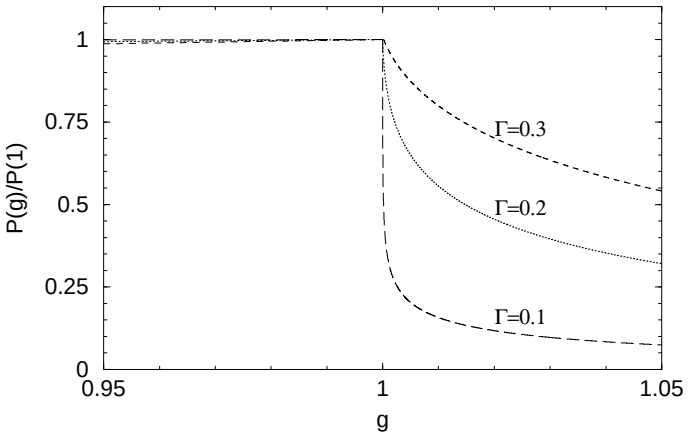

Fig. 1 – $P(g)/P(1)$ from Eq. (1) for different values of Γ.

In the above, we considered the case $\Gamma \ll 1$, where $x_3 \gg x_2 \gg 1$ and only the leading order terms were kept in the free energy. For weaker disorder but still in the insulating side, the next leading order terms will become important. First of all, the $\ln x_2$ term arising from $F^0(x_2, x_3)$ in (15) will start to contribute. Second, the next leading order contributions to F^0 will contribute some more x_2 and $\ln x_2$ terms. In general, as long as the limit $x_3 \gg x_2 \gg 1$ remains valid, and the perturbative expansion of the free energy remains a good approximation, the integral (18) can be generalized to be of the form

$$P(g) \propto \int_{\nu_1}^{x_{2max}} \mathrm{d}x_2 x_2^p e^{-a\Gamma x_2^2 + bx_2} \tag{19}$$

where a, b and p are parameters which in principle can be calculated within our framework by keeping higher order terms in the free energy. For $p = 0$ we get back (1) with renormalized parameters. For finite integer p the integral can again be evaluated simply in terms of the error function. For non-integer values of p, the integral is more complicated but it is clear that the qualitative features of the results will remain the same. We expect that our results will remain qualitatively valid near the crossover region on the insulating side ($\Gamma \lesssim 1/2$).

In order to check some of our approximate expressions, we have calculated $P(g)$ numerically for a quasi 1D wire. Recent techniques [4] allow us to obtain reasonable statistics for $N = 5$. Unfortunately, it was not possible to obtain enough data for either $\Gamma \ll 1$ or $1 < g < 1.01$. Nevertheless, (19) should be approximately valid in the region numerically accessible to us and the data can be used to check if the approximate scaling expected from (19) holds qualitatively. In Fig. 2, circles and squares show the numerical results for different values of average conductance $\bar{g}$. We use $p = 2$, $\Gamma_e = a\Gamma$ and b as fitting parameters to fit the data for different $\bar{g}$ using (19) (for our range of parameters, the constant C of (1) turns out to be negligible). Agreement with the data is good for $g - 1 < 0.15$. In Fig. 3 the same data are plotted as a function of the scaling variable $s = \exp[-\sqrt{\Gamma_e}(\nu - b/2\Gamma_e]$ shifted by $s(1 + \alpha)$, taking into account the fact that the singularities occur at $g = 1 + \alpha(\Gamma_e)$. Again the scaling is quite good. We have checked that the scaling fails for larger values of Γ_e, which is expected because it is close to the crossover regime ($\Gamma \sim 1/2$) where our perturbative expansion of the free energy starts to become invalid.

A non-analyticity in $P(g)$ was obtained in [4] within Random Matrix Theory (RMT), based

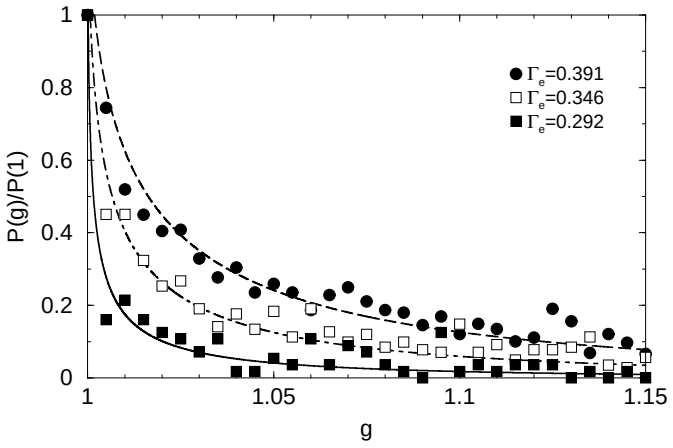

Fig. 2 – Numerical data for $N = 5$ with different disorder compared with theoretical predictions using (19). Data for $\bar{g} \approx 0.3$ were fit (solid line) with $\Gamma_e = a\Gamma = 0.292$ and $b = 2$; $\bar{g} \approx 0.4$ were fit (dot-dashed line) with $\Gamma_e = 0.346$ and $b = 1.8$; $\bar{g} \approx 0.5$ were fit (long dashed line) with $\Gamma_e = 0.391$ and $b = 1.6$. In all plots, $p = 2$.

on the assumption that contributions from only the smallest two eigenvalues are important in the insulating regime. All derivatives of $P(g)$ in such a model diverge at $g = 1$. However, a two-eigenvalue calculation within DMPK (obtained by neglecting the continuum in our model) gives rise to an essential singularity at $g = 1$, so that there is no discontinuity in the derivatives at that point. Therefore, it is not simple to see if the trivial discontinuity in the derivative mentioned in the introduction will persist beyond 1D. We emphasize that our present results for quasi 1D are fundamentally different from both of the above models: whereas all derivatives of P(g) *diverge* for the RMT model and *vanish* for the two-eigenvalue DMPK model, they are *finite* for our quasi 1D system. Also, in contrast to the above cases, the singularity for the quasi 1D system is not exactly at $g = 1$, but is shifted as a function of disorder. Finally, the RMT results do not satisfy the approximate scaling of Fig. 3.

We note that there is no phase transition in quasi 1D. It is therefore important to ask if a non-analyticity in $P(g)$ in the absence of a true phase transition violates any fundamental principle. Our framework can be considered as an electrostatic problem in one dimension involving charges with repulsive interactions and confinement potentials. Our free energy and the density of charges are all analytic, ensuring that e.g. the total energy will be an analytic function of disorder. However, the conductance is a complicated function of the charge distribution, and the non-analyticity appears only in the distribution of the conductances where the charges have to satisfy certain constraints. There is no restriction on the analyticity of such a distribution.

In summary, we have shown that the distribution of conductances in quasi 1D systems in the insulating regime has a non-analytic behavior near $g = 1$. The non-analyticity gives rise to very sharp structures close to $g = 1$, with finite discontinuities in its derivatives. The presence of similar structures in higher dimensions in numerically studied systems [6, 7, 8] gives rise to the possibility that such non-analyticity might be present in the conductance distribution in higher dimensions as well, having important consequences for the Anderson transition. Whether the non-analyticity discussed here disappears abruptly at some critical value of Γ, or smoothly, as Γ is increased beyond $1/2$ remains to be clarified. It is conceivable

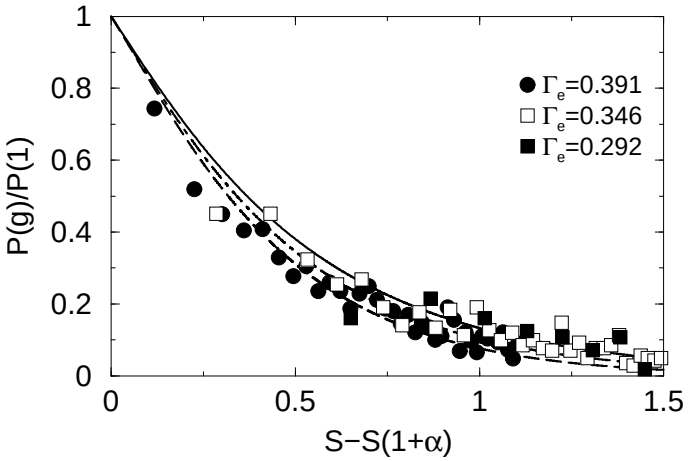

Fig. 3 – Appropriately scaled data from Fig. 2 showing approximate scaling as a function of $s = \exp[-\sqrt{\Gamma_e}(\nu - b/2\Gamma_e]$, shifted by $s(1+\alpha)$. The theoretical lines from Eq. (19) for different Γ_e (solid, dot-dashed and long-dashed lines for $\Gamma_e = 0.292$, 0.346 and 0.391, respectively) do not fall exactly on each other, showing the approximate nature of the scaling.

that an abrupt disappearance of the non-analyticity is a signature of the Anderson transition.

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